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Unitarity, quasi-normal modes and the $\mathbf{AdS_3/CFT_2}$ correspondence

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Abstract

In general, black-hole perturbations are governed by a discrete spectrum of complex eigenfrequencies (quasi-normal modes). This signals the breakdown of unitarity. In asymptotically AdS spaces, this is puzzling because the corresponding CFT is unitary. To address this issue in three dimensions, we replace the BTZ black hole by a wormhole, following a suggestion by Solodukhin [hep-th/0406130]. We solve the wave equation for a massive scalar field and find an equation for the poles of the propagator. This equation yields a rich spectrum of *real* eigenfrequencies. We show that the throat of the wormhole is $o(e^{-1/G})$, where G is Newton's constant. Thus, the quantum effects which might produce the wormhole are non-perturbative.

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Quasi-normal modes govern the time evolution of a perturbed black hole. Typically, they form a discrete spectrum of *complex* frequencies which are derived as the eigenvalues of a wave equation in the black-hole background. Their imaginary part is negative implying that the black hole eventually relaxes back to its original (thermal) equilibrium at (Hawking) temperature T_H [1]. This is due to leakage of information into the horizon and signals the breakdown of unitarity. It is closely related to Hawking's information loss paradox [2]. This issue is still unresolved [3–5]. Its resolution will require an understanding of quantum gravity beyond its semi-classical approximation.

In asymptotically AdS space-times, there is an additional tool in tackling the problem of non-unitarity, due to the AdS/CFT correspondence [6]. The complex quasi-normal frequencies are poles of the retarded propagator in the corresponding CFT living on the boundary of AdS space. This is puzzling, because the CFT is unitary and therefore the propagator should possess *real* poles only. On the gravity side, the Poincaré recurrence theorem implies that, e.g., a two-point function would be quasi-periodic with a period t_P, say. For times $t \ll t_P$, the system may look like it is decaying back to thermal equilibrium, but for $t \gtrsim t_P$, it should return to its original state (or close) an infinite number of times. In fact, the theorem guarantees that the system will *never* relax back to its original state.

This problem has been tackled in three dimensions, where exact results can be derived [7–9]. AdS_3 arises in type IIB superstring theory in the near horizon limit of a large number of D1 and D5 branes [6]. Low energy excitations form a gas of strings wound around a circle with winding number $\mathtt{k}$ and target space T^4. For simplicity, we set the radius of the circle equal to one (unit circle). They are described by a strongly coupled CFT_2 whose central charge is

$$c = 6\mathtt{k} \sim \frac{1}{G} \gg 1 \tag{1}$$

where G is the three-dimensional Newton's constant. At finite temperature, the thermal CFT_2 has entropy

$$S \sim \mathtt{k} \sim \frac{1}{G} \tag{2}$$

On the gravity side, the finite temperature is provided by a BTZ black hole [10]. If the radius of its horizon is $o(1)$, then so is the area of the horizon ($A \sim 1$) and the Bekenstein-Hawking

entropy is

$$S = \frac{A}{4G} \sim \frac{1}{G} \tag{3}$$

in agreement with the CFT result (2). For such a system, the Poincaré recurrence time is estimated to be [11]

$$t_P \sim e^S \sim o(e^{1/G}) \tag{4}$$

It is clear from this expression that in order to understand the gravity side of the AdS/CFT correspondence at finite $G \sim 1/\mathtt{k}$, one ought to include contributions to gravity correlators beyond the semi-classical approximation which will modify the BTZ black-hole background. A number of alternatives have been entertained [11–17].

Here we follow a proposal by Solodukhin [17] and replace the black hole by a wormhole, thus eliminating the horizon and the attendant leakage of information. The size of the narrow throat will be

$$\lambda \sim o(e^{-1/G}) \tag{5}$$

leading to a Poincaré recurrence time

$$t_P \sim \frac{1}{\lambda} \sim o(e^{1/G}) \tag{6}$$

in agreement with expectations (eq. (4)). We shall calculate two-point functions explicitly and obtain the *real* poles of the propagator, thus demonstrating unitarity.

We start by reviewing known exact results for the BTZ black hole [7–9]. The metric for a non-rotating BTZ black hole reads

$$ds^2 = -\sinh^2 y \, dt^2 + dy^2 + \cosh^2 y \, d\phi^2 \tag{7}$$

where $\phi \in [0, 2\pi)$. The horizon is at $y = 0$ and $T_H = \frac{1}{2\pi}$ is the Hawking temperature. The wave equation for a massive scalar of mass m is

$$\frac{1}{\sinh y \cosh y} \frac{\partial}{\partial y} \left(\sinh y \cosh y \frac{\partial \Phi}{\partial y} \right) - \frac{1}{\sinh^2 y} \frac{\partial^2 \Phi}{\partial t^2} + \frac{1}{\cosh^2 y} \frac{\partial^2 \Phi}{\partial \phi^2} = m^2 \Phi \tag{8}$$

to be solved outside the horizon ($y > 0$). The solution may be written as

$$\Phi = e^{i(\omega t - k\phi)} \Psi(y), \qquad k \in \mathbb{Z} \tag{9}$$

where Ψ satisfies

$$\frac{1}{\sinh y\cosh y}\ (\sinh y\cosh y\ \Psi')' + \left(\frac{\omega^2}{\sinh^2 y}+\frac{k^2}{\cosh^2 y}\right)\Psi = m^2\Psi \tag{10}$$

Two independent solutions are obtained by examining the behavior near the horizon ($y \to 0$),

$$\Psi_\pm \sim y^{\pm i\omega} \tag{11}$$

They can be written in closed form in terms of hypergeometric functions,

$$\Psi_\pm = \cosh^{-2h_+} y\ \ \tanh^{\pm i\omega} y\ F(h_+ \pm \tfrac{i}{2}(\omega+k), h_+ \pm \tfrac{i}{2}(\omega-k); 1\pm i\omega; \tanh^2 y) \tag{12}$$

where $h_\pm = \frac{1}{2}(1\pm\sqrt{1+m^2})$. The acceptable solution is Ψ_- (purely in-going at the horizon). Near the boundary ($y \to \infty$), it behaves as

$$\Psi_- \sim \mathcal{A}_+\ e^{-2h_- y} + \mathcal{A}_-\ e^{-2h_+ y}\quad,\quad \mathcal{A}_\pm = \frac{\Gamma(2h_\pm - 1)\Gamma(1-i\omega)}{\Gamma(h_\pm - \frac{i}{2}(\omega+k))\Gamma(h_\pm - \frac{i}{2}(\omega-k))} \tag{13}$$

For quasi-normal modes, we demand that Ψ_- vanish at the boundary and therefore set $\mathcal{A}_+ = 0$. This leads to a discrete spectrum of complex frequencies,

$$\omega_n = \pm k - i(n+h_+)\quad,\quad n = 0,1,2,\ldots \tag{14}$$

with negative imaginary part, as expected [1]. It is perhaps worth mentioning that these frequencies may also be obtained by a simple monodromy argument which makes use of the *unphysical* black-hole singularity [18].

Turning to the AdS/CFT correspondence, the flux at the boundary ($y \to \infty$) is related to the retarded propagator of the corresponding CFT living on the boundary. A standard calculation yields

$$\tilde{G}_R(\omega,k) \sim \lim_{y\to\infty}\frac{F'(y)}{F(y)} \tag{15}$$

Explicitly,

$$\begin{aligned}\tilde{G}_R(\omega,k) &\sim \frac{\mathcal{A}_-}{\mathcal{A}_+}\\ &\sim \frac{\Gamma(h_+ - \frac{i}{2}(\omega-k))\Gamma(h_+ - \frac{i}{2}(\omega+k))}{\Gamma(h_- - \frac{i}{2}(\omega-k))\Gamma(h_- - \frac{i}{2}(\omega+k))}\\ &\sim |\Gamma(h_+ - \tfrac{i}{2}(\omega-k))\Gamma(h_+ - \tfrac{i}{2}(\omega+k))|^2\\ &\quad \times \sin\pi(h_+ - \tfrac{i}{2}(\omega-k))\sin\pi(h_+ - \tfrac{i}{2}(\omega+k))\end{aligned} \tag{16}$$

Plainly, the quasi-normal modes (zeroes of $\mathcal{A}_+$) are poles of the retarded propagator (since $\tilde{G}_R \sim 1/\mathcal{A}_+$). Its fourier transform has an exponentially decaying tail

$$G_R(t) = \int d\omega \; e^{-i\omega t} \tilde{G}_R \sim e^{-h_+ t} \tag{17}$$

as $t \to \infty$, exhibiting no Poincaré recurrences. It was pointed out [11] that this does not contradict the unitarity of the corresponding CFT, because the latter effectively lives in infinite space. This is because a string with winding number $\mathtt{k}$ sees a space of length

$$L_{eff} \sim \mathtt{k} \tag{18}$$

The AdS/CFT correspondence for a BTZ black hole only works in the infinite-$\mathtt{k}$ limit. To consider the large but finite $\mathtt{k}$ case, we turn to a discussion of the AdS/CFT correspondence for a wormhole.

The wormhole metric is [17]

$$ds^2 = -(\sinh^2 y + \lambda^2) \; dt^2 + dy^2 + \cosh^2 y \; d\phi^2 \tag{19}$$

In the limit $\lambda \to 0$, it reduces to the BTZ black hole metric (7). There is no horizon at $y = 0$; instead the wormhole has a very narrow throat ($o(\lambda)$) joining two regions of space-time with two distinct boundaries (at $y \to \pm\infty$, respectively). Information may flow in both directions through the throat. This modification is significant near the "horizon" point $y = 0$. As we approach $y \to 0$, the time-like distance is $ds^2 \approx -\lambda^2 dt^2$, showing that the time scale of the system is $\sim 1/\lambda$. This is the order of magnitude of the Poincaré recurrence time, i.e., we expect

$$t_P \sim o(1/\lambda) \tag{20}$$

as advertised (eq. (6)). The value of λ will be fixed upon comparison with the dual CFT.

The radial wave equation for a massive scalar of mass m is now

$$\frac{1}{\cosh y \; (\sinh^2 y + \lambda^2)^{1/2}} \left(\cosh y \; (\sinh^2 y + \lambda^2)^{1/2} \; \Psi'\right)' + \left(\frac{\omega^2}{\sinh^2 y + \lambda^2} + \frac{k^2}{\cosh^2 y}\right) \; \Psi = m^2 \Psi \tag{21}$$

to be solved along the entire real axis ($y \in \mathbb{R}$), unlike in the black hole case (eq. (8)), where y was restricted to positive values, since the horizon was at $y = 0$. We wish to solve this equation in the small-λ limit ($\lambda \ll 1$). To this end, we consider three regions,

(I) $y \gg \lambda$ which includes one of the boundaries,

(II) $y \ll -\lambda$ which includes the other boundary, and

(III) $|y| \ll 1$.

We shall solve the wave equation in each region and then match the solutions in the overlapping regions $\lambda \ll y \ll 1$ and $-1 \ll y \ll -\lambda$. Let us start with region (II). In this region, the wave equation may be approximated by the corresponding equation for the BTZ black hole (10). Two independent solutions are then given by (12). However, in our case there is no physical requirement dictating a choice based on the small-y behavior, because there is no horizon at $y = 0$. In fact, we ought to chose a linear combination which behaves nicely at the boundary ($y \to -\infty$). Thus, the acceptable solution in this region is

$$\Psi_{II} = \cosh^{-2h_+} y \;\; \tanh^{-i\omega} y \;\, F(h_+ - \tfrac{i}{2}(\omega + k), h_+ - \tfrac{i}{2}(\omega - k); 2h_+; 1/\cosh^2 y) \qquad (22)$$

It vanishes at the boundary ($\Psi_{II} \sim e^{2h_+ y}$ as $y \to -\infty$). At small y, it behaves as

$$\Psi_{II} \sim \mathcal{B}_+ y^{-i\omega} + \mathcal{B}_- y^{+i\omega} \quad , \quad \mathcal{B}_\pm = \frac{\Gamma(2h_+)\Gamma(\pm i\omega)}{\Gamma(h_+ \pm \frac{i}{2}(\omega + k))\Gamma(h_+ \pm \frac{i}{2}(\omega - k))} \qquad (23)$$

In region (III) ($|y| \ll 1$), the wave equation (21) reduces to

$$\frac{1}{(y^2 + \lambda^2)^{1/2}} \left((y^2 + \lambda^2)^{1/2} \; \Psi'_{III} \right)' + \frac{\omega^2}{y^2 + \lambda^2} \; \Psi_{III} = 0 \qquad (24)$$

Two linearly independent solutions are

$$\Psi_{III}^{(1)} = F(\tfrac{i}{2}\omega, -\tfrac{i}{2}\omega; \tfrac{1}{2}; -y^2/\lambda^2) \quad , \quad \Psi_{III}^{(2)} = \frac{y}{\lambda} \, F(\tfrac{1}{2} + \tfrac{i}{2}\omega, \tfrac{1}{2} - \tfrac{i}{2}\omega; \tfrac{1}{3}; -y^2/\lambda^2) \qquad (25)$$

At large $y > 0$, they behave as

$$\Psi_{III}^{(1)} \sim \frac{1}{2} \left(\frac{2y}{\lambda} \right)^{+i\omega} + \frac{1}{2} \left(\frac{2y}{\lambda} \right)^{-i\omega} \quad , \quad \Psi_{III}^{(2)} \sim \frac{i}{2\omega} \left(\frac{2y}{\lambda} \right)^{+i\omega} - \frac{i}{2\omega} \left(\frac{2y}{\lambda} \right)^{-i\omega} \qquad (26)$$

Analytically continuing to negative y, we note that $\Psi_{III}^{(1)}$ is an even function whereas $\Psi_{III}^{(2)}$ is odd. We therefore obtain the asymptotic behavior

$$\Psi_{III}^{(1)} \sim \frac{1}{2} \left(\frac{2y}{\lambda} \right)^{+i\omega} + \frac{1}{2} \left(\frac{2y}{\lambda} \right)^{-i\omega} \quad , \quad \Psi_{III}^{(2)} \sim -\frac{i}{2\omega} \left(\frac{2y}{\lambda} \right)^{+i\omega} + \frac{i}{2\omega} \left(\frac{2y}{\lambda} \right)^{-i\omega} \qquad (27)$$

as $y \to -\infty$. Matching this to the asymptotic behavior of Ψ_{II} given by eq. (23), we obtain the acceptable solution in region (III),

$$\Psi_{III} = \mathcal{B}_+ \Psi_{III}^{(-)} + \mathcal{B}_- \Psi_{III}^{(+)} \quad , \quad \Psi_{III}^{(\pm)} = \left(\frac{2}{\lambda}\right)^{\mp i\omega} \left(\Psi_{III}^{(1)} \pm i\omega \Psi_{III}^{(2)}\right) \tag{28}$$

On account of (26), at large $y > 0$, it behaves as (*cf.* eq. (23))

$$\Psi_{III} \sim \mathcal{B}_+ \left(\frac{4y}{\lambda^2}\right)^{+i\omega} + B_- \left(\frac{4y}{\lambda^2}\right)^{-i\omega} \tag{29}$$

Finally, in region (I), two linearly independent solutions are (*cf.* eq. (22) in region (II))

$$\Psi_I^{(\pm)} = \cosh^{-2h_\pm} y \;\; \tanh^{-i\omega} y \; F(h_\pm - \tfrac{i}{2}(\omega + k), h_\pm - \tfrac{i}{2}(\omega - k); 2h_\pm; 1/\cosh^2 y) \tag{30}$$

Matching the asymptotic behavior (29), we obtain the solution in region (I),

$$\Psi_I = \alpha_+ \Psi_I^{(-)} + \alpha_- \Psi_I^{(+)} \tag{31}$$

where

$$\alpha_+ = \frac{\mathcal{B}_+^2 \left(\frac{2}{\lambda}\right)^{2i\omega} - \mathcal{B}_-^2 \left(\frac{2}{\lambda}\right)^{-2i\omega}}{\mathcal{B}_+\mathcal{C}_- - \mathcal{B}_-\mathcal{C}_+} \quad , \quad \alpha_- = \frac{\mathcal{B}_-\mathcal{C}_- \left(\frac{2}{\lambda}\right)^{-2i\omega} - \mathcal{B}_+\mathcal{C}_+ \left(\frac{2}{\lambda}\right)^{2i\omega}}{\mathcal{B}_+\mathcal{C}_- - \mathcal{B}_-\mathcal{C}_+} \tag{32}$$

$\mathcal{B}_\pm$ are given by (23) and

$$\mathcal{C}_\pm = \frac{\Gamma(2h_-)\Gamma(\pm i\omega)}{\Gamma(h_- \pm \frac{i}{2}(\omega + k))\Gamma(h_- \pm \frac{i}{2}(\omega - k))} \tag{33}$$

For a normalizable solution, we ought to set $\alpha_+ = 0$, which leads to the quantization condition

$$\left(\frac{2}{\lambda}\right)^{2i\omega} = \frac{\mathcal{B}_-}{\mathcal{B}_+} = \frac{\Gamma(-i\omega)\Gamma(h_+ + \frac{i}{2}(\omega + k))\Gamma(h_+ + \frac{i}{2}(\omega - k))}{\Gamma(+i\omega)\Gamma(h_+ - \frac{i}{2}(\omega + k))\Gamma(h_+ - \frac{i}{2}(\omega - k))} \tag{34}$$

This leads to a discrete spectrum of *real* frequencies (notice that both sides of the equation have unit norm, since $\mathcal{B}_-$ is the complex conjugate of $\mathcal{B}_+$ for real ω). For small ω, we may approximate $\mathcal{B}_-/\mathcal{B}_+ \approx -1$, therefore we obtain the spectrum

$$\omega_n \approx \left(n + \tfrac{1}{2}\right) \frac{\pi}{\ln \frac{2}{\lambda}} \quad , \quad n \in \mathbb{Z} \tag{35}$$

corresponding to periodicity with period $L_{eff} \sim \ln(1/\lambda)$ [11]. Comparing with the CFT result that the periodicity is $L_{eff} \sim \mathtt{k} \sim 1/G$ (eq. (18)), we deduce

$$\lambda \sim o(e^{-1/G}) \tag{36}$$

as promised (eq. (5)), showing that the wormhole is produced by non-perturbative quantum effects. Notice also that the period of the CFT propagator L_{eff} is much smaller than the Poincaré recurrence time (eq. (4)),

$$L_{eff} \ll t_P \tag{37}$$

This is reminiscent of the chaotic behavior of the brick wall modification of the BTZ black hole [17] and is attributed to the rich structure of the spectrum given by (34).

The retarded propagator is

$$\tilde{G}_R(\omega, k) \sim \frac{\alpha_-}{\alpha_+} \sim \frac{\mathcal{B}_-\mathcal{C}_- - \mathcal{B}_+\mathcal{C}_+ \left(\frac{2}{\lambda}\right)^{4i\omega}}{\mathcal{B}_-^2 - \mathcal{B}_+^2 \left(\frac{2}{\lambda}\right)^{4i\omega}} \tag{38}$$

and has real poles given by (34). It is defined in the upper-half complex ω-plane. In the limit $\lambda \to 0$ (or, equivalently, $\mathtt{k} \to \infty$), the spectrum of real frequencies (35) becomes continuous, signaling the emergence of a horizon. The retarded propagator becomes

$$\tilde{G}_R(\omega, k) \sim \frac{\mathcal{C}_-}{\mathcal{B}_-} \tag{39}$$

which agrees with our earlier BTZ result (eq. (16)), since $\mathcal{B}_- \sim \mathcal{A}_+$ and $\mathcal{C}_- \sim \mathcal{A}_-$ (using eqs. (13), (23) and (33)). In this limit, the quasi-normal modes (14) emerge as poles (zeroes of $\mathcal{B}_- \sim \mathcal{A}_+$). It should be emphasized that for no other value of λ, no matter how small, do complex poles arise.

To summarize, we replaced the BTZ black hole with a wormhole [17] in order to study the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence at finite temperature and large but finite central charge (1) of the CFT. The wormhole had a narrow throat of size λ. We argued that the system, once perturbed, exhibited Poincaré recurrences with time constant $t_P \sim 1/\lambda$. We solved the wave equation for a scalar field in the small-λ limit and deduced the propagator for the dual CFT. We found an explicit equation for the poles (eq. (34)) which yielded a rich spectrum of *real* eigenvalues demonstrating the unitarity of time evolution. Upon comparison with the expected periodicity of the dual CFT, we deduced that the wormhole parameter λ was of order $e^{-1/G}$, suggesting that the wormhole can only emerge through non-perturbative effects.

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